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ON OPTIMAL ECONOMIC GROWTH WITH VARIABLE DISCOUNT RATES: EXISTENCE AND STABILITY RESULTS*

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1. INTRODUCTION

In this paper, I shall consider a framework of optimal growth, where the technology is specified by the well-known one-good neoclassical model, and the planner's objective function is of the form:

(1) $$\sum_{t=1}^{T} \alpha_t u(c_t) \quad \text{where} \quad \alpha_t > 0, \quad \text{for} \quad t \geq 1,\ \alpha_1 = 1.$$

For $t \geq 1$, $\beta_{t+1} = (\alpha_{t+1}/\alpha_t)$ is the *discount factor* for time period $(t+1)$, and $\gamma_{t+1} = [(1/\beta_{t+1}) - 1]$ is the corresponding *discount rate*. In contrast to the traditional objective functions used in Cass [1965] and Koopmans [1965], we shall allow the discount factor (and, hence, the discount rate) to vary over time.

If the planning horizon, T, is infinitely large, and we allow the sequence of discount factors to vary arbitrarily, it is clear that an optimal (or a weakly-maximal program) will not, in general exist. (Optimality and Weak Maximality are defined, following the approach of Gale [1967] and Brock [1970], in Section 2). On the other hand, if the sequence of discount factors satisfies certain conditions (e.g., the discount factor is constant and <1), then we know that an optimal program will exist. A natural question that arises then is the following: Is there some easily applicable criterion by which the two cases can be distinguished? In other words, is there a necessary and sufficient condition on the discount factors for the existence of optimal (or weakly maximal) programs?

This question is answered in the paper, in Theorems 1 and 2. Weakly maximal programs are shown to exist if and only if $\sum_{t=1}^{T} (1/\alpha_t) \to \infty$ as $T \to \infty$. (Theorem 1) This means, in particular, that even when future utilities are given greater weights than current ones, weakly maximal programs will exist, provided such weights do not become large "too fast".

Optimal programs will exist if and only if there is $A < \infty$, such that $\alpha_t \leq A$ for $t \leq 1$ (Theorem 2). This characterization compared to the one on weakly-maximal programs, indicates the difference between optimality and weak-maximality precisely. Thus, if $\alpha_t \to \infty$, weak-maximal programs could exist, but

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optimal programs could not.

Another question of long-standing interest in this area is the following. When weakly-maximal (or optimal) programs *do* exist, what can we say regarding their long-run behavior? In other words, is there some sense in which such programs are asymptotically stable? This is answered in Theorem 3, where weakly-maximal programs from different initial stocks are shown to exhibit a "twisted turnpike" property; that is, they converge to each other. It should be noted that this is really the basic turnpike property. The more familiar result, where optimal programs are shown to converge to an optimal stationary program ("golden rule") is a consequence of the assumption of stationary intertemporal preferences.

There is very little in the literature on optimal growth theory with variable discount factors which deals with the above-mentioned results. Brock [1971] examined some relations between "sensitivity results" and the existence of weakly-maximal programs. Similar results also appear in Mirrlees and Hammond [1973] in their study on "agreeable plans." However, no existence result like the one proved here, was established. It should be observed, however, that it is particularly the results of Brock [1971], and the result on the complete characterization of efficiency by Cass [1972] that enables one to arrive at Theorem 1, as will be evident from the technique of proof. It should be mentioned, in this connection, that, since the results of Brock [1971] have not been generalized to the multi-sector case, the generality of the results of this paper remains an open question.

The existence of an optimal program is often established in the literature (see, particularly, Weizsacker [1965], and McKenzie [1974] for the case where tastes are allowed to vary over time), by assuming that there is a competitive program,[2] for which the value of input, at the competitive prices, is bounded above, and showing, then, under certain conditions, that this competitive program is optimal. This is closely related to the sufficiency part of Theorem 2 (even though we prove what is assumed in these papers). However, the *necessity* part of Theorem 2 is completely absent in their works.[3]

On the asymptotic behavior of weakly-maximal programs, we note that McKenzie [1976] proves a similar result in a much more general context, but by assuming properties on weakly maximal programs, like "uniform reachability," which are, in general, difficult to verify. Our result is proved without any such assumptions, and is useful because it provides a standard by which generalizations may be evaluated.

It should be mentioned that optimizing models with variable discount rates have been examined by Strotz [1955–56], Pollak [1968], and Peleg and Yaari [1973], to investigate the problem of myopia and inconsistency in planning; and by Uzawa [1968] to investigate some qualitative properties of short and long-run

[2] See Section 2 for a definition of this concept.

[3] This, however, does not mean that the necessity part has not received any attention in the literature. See especially McKenzie [1976] for a discussion of the difficulties of proving the necessity result in a general model.

consumption functions. I should state that these interesting studies are concerned with issues which are quite different from those examined in this paper.

2. THE MODEL

We consider a neoclassical one-good model, with a technology given by a function f from R^+ to itself. The production possibilities consist of inputs, x, and outputs, $y=f(x)$ for $x\geq 0$. The following assumptions on $f(\cdot)$ are maintained throughout the paper:

(F.1) $f(0)=0$.

(F.2) $f(x)$ is strictly increasing for $x\geq 0$.

(F.3) $f(x)$ is continuous for $x\geq 0$, and twice continuously differentiable for $x>0$.

(F.4) $f(x)$ is strictly concave for $x\geq 0$, and $f''(x)<0$ for $x>0$.

(F.5) $f(x)$ satisfies the end-point conditions: $f'(x)\to d<1$, as $x\to\infty$; $f'(x)\to\infty$ as $x\to 0$.

We define a *feasible production program* from $\underset{\sim}{x}>0$, as a sequence $(x, y) = (x_t, y_{t+1})$ satisfying

$$(2)\qquad x_0=\underset{\sim}{x},\ 0\leq x_t\leq y_t \quad \text{for} \quad t\geq 1, \quad \text{and} \quad f(x_t)=y_{t+1} \quad \text{for} \quad t\geq 0.$$

The *consumption program* $c=(c_t)$, generated by (x, y) is given by

$$(3)\qquad c_t = y_t - x_t\ (\geq 0) \qquad \text{for} \quad t\geq 1.$$

We shall refer to (x, y, c) as a *feasible program*, it being understood that (x, y) is a production program, and c is the corresponding consumption program.

A feasible program (x, y, c) from $\underset{\sim}{x}>0$, *dominates* a feasible program (x^*, y^*, c^*) from $\underset{\sim}{x}$, if $c_t\geq c_t^*$ for all $t\geq 1$, and $c_t>c_t^*$ for some t. A feasible program (x^*, y^*, c^*) from $\underset{\sim}{x}$ is said to be *inefficient* if some feasible program from $\underset{\sim}{x}$ dominates it. An *efficient program* is a feasible program which is not inefficient.

Under (F.1)–(F.5), there exist unique numbers $\hat{b}$, b satisfying $0<\hat{b}<b<\infty$, and $f'(\hat{b})=1, f(b)=b$. For any feasible program (x, y, c) from $\underset{\sim}{x}\in(0, b)$, we have $x_t, y_{t+1}, c_{t+1}\leq b$ for $t\geq 0$. ($(0, b)$ represents the open interval $\{z\in R^+: 0<z<b\}$). A feasible program (x, y, c) from $\underset{\sim}{x}>0$, is called *interior* if $x_t, c_{t+1}>0$, for $t\geq 0$. For an interior program (x, y, c), we denote by π_t the expression $\prod_{s=0}^{t-1} f'(x_s)$, for $t\geq 1$.

The planner is endowed with a utility function u (defined on the non-negative reals) and a sequence $\alpha=(\alpha_t)$ of positive numbers, which reflect the planner's time preference. Following Brock [1970], a feasible program (x^*, y^*, c^*) from $\underset{\sim}{x}>0$, is called *weakly-maximal* if

$$(4)\qquad \liminf_{T\to\infty} \sum_{t=1}^{T} \alpha_t[u(c_t) - u(c_t^*)] \leq 0$$

for every feasible program (x, y, c) from $\underset{\sim}{x}$. Similarly, following Gale [1967] a feasible program (x^*, y^*, c^*) from $\underset{\sim}{x}>0$, is called *optimal* if

(5) $$\limsup_{T\to\infty} \sum_{t=1}^{T} \alpha_t[u(c_t) - u(c_t^*)] \le 0$$

for every feasible program (x, y, c) from $\underset{\sim}{x}$.

A feasible program (x^*, y^*, c^*) from $\underset{\sim}{x}>0$, is called *competitive*[4] if there is a non-null sequence $p^*=(p_t^*)$ of non-negative prices, such that

(6) $$\alpha_t u(c_t^*) - p_t^* c_t^* \ge \alpha_t u(c) - p_t^* c, \qquad c \ge 0,\ t \ge 1$$

(7) $$p_{t+1}^* y_{t+1}^* - p_t^* x_t^* \ge p_{t+1}^* y - p_t^* x \qquad \text{for} \quad x \ge 0,\ y = f(x),\ t \ge 0.$$

A price sequence $p^*=(p_t^*)$, associated with a competitive program (x^*, y^*, c^*), for which (6) and (7) hold, is called a sequence of *competitive prices*; (6) and (7) are called *competitive conditions*.

The following assumptions on u will be used in this paper:

(U.1) $u(c)$ is strictly increasing for $c \ge 0$.

(U.2) $u(c)$ is twice continuously differentiable for $c > 0$.

(U.3) $u(c)$ is concave for $c \ge 0$ ($u''(c) \le 0$ for $c > 0$).

(U.4) $u(c)$ satisfies the end-point condition: $u'(c)\to\infty$ as $c\to 0$.

The sequence $\alpha=(\alpha_t)$ will be assumed to satisfy:

(A.1) $\alpha_1 = 1$; and, there are positive numbers m, $\hat{m}$ such that

$$\hat{m} \le (\alpha_{t+1}/\alpha_t) f'(b) \le (1 - m) \qquad \text{for} \quad t \ge 1.$$

We note here that, under the assumptions, an interior program (x^*, y^*, c^*) is competitive iff the following *Euler conditions* hold:

(8) $$[\alpha_t u'(c_t^*)]/[\alpha_{t+1} u'(c_{t+1}^*)] = f'(x_t^*) \qquad \text{for} \quad t \ge 1.$$

This result is easily established, and is stated here for ready reference, as it is often used in the next three sections. When an interior program (x^*, y^*, c^*) is competitive, then the sequence of competitive prices $p^*=(p_t^*)$ are seen to be given by:

(9) $$p_0^* = \alpha_1 u'(c_1^*) f'(x_0^*);\ p_t^* = \alpha_t u'(c_t^*) = p_0^*/\pi_t^* \qquad \text{for} \quad t \ge 1.$$

3. A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF WEAKLY MAXIMAL PROGRAMS

In this section, I shall establish that a weakly-maximal program exists if and only if the discount factors satisfy the condition that $\sum_{t=1}^{T} (1/\alpha_t)\to\infty$, as $T\to\infty$. This result, of course, includes as a special case the fact that if the discount factor is constant (say, $\beta>0$), then a weakly-maximal program exists if and only if

[4] This definition was introduced by Gale and Sutherland [1968], and is a modified version of the definition used in Gale [1967].

$\beta \leq 1$. However, the result also points out the possibility that weakly-maximal programs could exist when the discount factor is variable, and, at each point of time, is greater than unity. Thus if t^* is the smallest integer such that $(1/t^*) < [1/f'(b)] - 1$, and $\alpha_t = t^* + t$, for $t \geq 2$, then the variable discount factor at each point of time is greater than unity (i.e., the variable discount rate at each point of time is negative), but since $\sum_{t=1}^{T}(1/\alpha_t) \to \infty$ as $T \to \infty$, a weakly maximal program will exist.

THEOREM 1. *Under (F.1)–(F.5), (U.1)–(U.4), and (A.1), a weakly maximal program exists from* $\underset{\sim}{x} \in (0, b)$, *if and only if*

$$\sum_{t=1}^{T}(1/\alpha_t) \longrightarrow \infty \quad as \quad T \longrightarrow \infty. \tag{10}$$

PROOF. (*Necessity*) Suppose a weakly maximal program exists from $\underset{\sim}{x} \in (0, b)$, call it $(\bar{x}, \bar{y}, \bar{c})$. Then, for each $t \geq 1$, the expression $\alpha_t u[f(\bar{x}_{t-1}) - x] + \alpha_{t+1} u[f(x) - \bar{x}_{t+1}]$ must be a maximum at $x = \bar{x}_t$. By (U.4), the maximum must be at an interior point, i.e., $\bar{c}_t > 0$ for $t \geq 1$. Hence, $\bar{x}_t > 0$ for $t \geq 0$, so that for each $t \geq 1$, $\alpha_t u'(\bar{c}_t)(-1) + \alpha_{t+1} u'(\bar{c}_{t+1}) f'(\bar{x}_t) = 0$; or, by transposition,

$$[\alpha_t u'(\bar{c}_t)]/[\alpha_{t+1} u'(\bar{c}_{t+1})] = f'(\bar{x}_t) \qquad \text{for} \quad t \geq 1. \tag{11}$$

Using concavity of u and f, and (11), (x, y, c) is competitive.

Using (11) repeatedly, we have for $T \geq 2$,

$$[\alpha_1 u'(\bar{c}_1)]/[\alpha_T u'(\bar{c}_T)] = \bar{\pi}_T / f'(\underset{\sim}{x}). \tag{12}$$

We claim now that

$$\sum_{T=2}^{\tau} \bar{\pi}_T \longrightarrow \infty \quad \text{as} \quad \tau \longrightarrow \infty. \tag{13}$$

In order to establish our claim, we need two lemmas. The logic of the lemmas in the proof of the theorem should, perhaps, be explained. We know that since $(\bar{x}, \bar{y}, \bar{c})$ is weakly-maximal, it is also efficient. The theorem of Cass [1972] on efficiency [if $(\bar{x}, \bar{y}, \bar{c})$ is efficient, then $\sum_{t=0}^{T} \bar{\pi}_t \to \infty$ as $T \to \infty$] can then be applied to obtain (13), provided we know that $\inf_{t \geq 0} \bar{x}_t > 0$.

However, if $\inf_{t \geq 0} \bar{x}_t = 0$, then this theorem cannot be applied, and we have to devise an alternative route. We note that, if $\lim_{t \to \infty} \bar{x}_t = 0$, then we can directly check that (13) is satisfied. So, the only case which can present problems in verifying (13), is where there is one subsequence of $\bar{x}_t$ which converges to zero, and another subsequence of $\bar{x}_t$ which remains bounded away from zero.[5] Lemma 2 shows that, given the structure of the model, this problematic case can never occur. This

[5] A non-negative sequence (w_t) is said to be *bounded away from zero* if $\inf_{t \geq 0} w_t > 0$. It is said to be *bounded above* if $\sup_{t \geq 0} w_t < \infty$.

is proved with the help of Lemma 1, which shows that an interior competitive program satisfied a "catenary property" *for "low" input levels*, viz., if input levels are falling at a certain stage, they must be falling at all succeeding stages.

Let us define $\hat{x}$, such that $f'(\hat{x})=2f'(b)/\hat{m}$, and $0<\hat{x}<b$. By (F.5), this is possible. We can now state and prove (under the Theorem's assumptions),

LEMMA 1. *If a feasible program* (x, y, c) *from* $\underset{\sim}{x}\in(0, b)$ *is interior and competitive, and there is* $t'\geq 1$, *such that* (i) $x_{t'}\leq\hat{x}$, *and* (ii) $x_{t'-1}>x_{t'}$, *then* (x_t) *is a convergent sequence.*

PROOF OF LEMMA 1. Note that, since (x, y, c) is interior and competitive, so $u'(c_{t'})=(\alpha_{t'+1}/\alpha_{t'})f'(x_{t'})u'(c_{t'+1})>u'(c_{t'+1})$ by using (i), the definition of $\hat{x}$, and (A.1). Hence, $c_{t'}<c_{t'+1}$, i.e., $f(x_{t'-1})-x_{t'}<f(x_{t'})-x_{t'+1}$. Hence, using (ii) in this inequality, we have $x_{t'+1}<x_{t'}$. We have, thus, established that (i') $x_{t'+1}\leq\hat{x}$ and (ii') $x_{t'}>x_{t'+1}$. Hence, the above argument can be repeated for all succeeding time periods to establish that (x_t) is monotonically decreasing for $t\geq t'-1$. Since (x_t) is bounded below by zero, so it is a convergent sequence.

LEMMA 2. *If a feasible program* (x, y, c) *from* $\underset{\sim}{x}\in(0, b)$ *is interior and competitive, then either* (i) $x_t\to 0$ *as* $t\to\infty$, *or* (ii) $\inf_{t\geq 0} x_t>0$.

PROOF OF LEMMA 2. Suppose, contrary to the Lemma, that there is one subsequence of (x_t) converging to zero, and another subsequence of (x_t) which remains bounded away from zero. Then there is some $t'>1$, such that $x_{t'}\leq\min\left(\hat{x}, \frac{1}{2}\underset{\sim}{x}\right)$. Now, clearly, it is not possible that $x_{t'-1}>x_{t'}$. For, then, by Lemma 1, (x_t) would be a convergent sequence. Hence $x_{t'-1}\leq x_{t'}$. But, then, $x_{t'-1}\leq\min\left(\hat{x}, \frac{1}{2}\underset{\sim}{x}\right)$, so that the above argument can be repeated to establish that $x_{t'-2}\leq x_{t'-1}$. Similarly, the argument could be repeated for all preceding time periods, yielding $\underset{\sim}{x}\leq x_1\leq\cdots\leq x_{t'-1}\leq x_{t'}$, which contradicts the fact that $x_{t'}\leq\frac{1}{2}\underset{\sim}{x}$, and proves the Lemma.

We return, now, to the proof of Theorem 1. By Lemma 2, either (a) $\bar{x}_t\to 0$ as $t\to\infty$, or (b) $\inf_{t\geq 0}\bar{x}_t>0$. In case (a), there is t^*, such that for $t\geq t^*$, $f'(\bar{x}_t)\geq 1$, which establishes (13). In case (b), noting that $(\bar{x}, \bar{y}, \bar{c})$ is efficient, (13) follows from Theorem 3 in Cass [1972]. Thus, in either case, claim (13) is established.

Now, note that for $T\geq 2$, $\bar{c}_T\leq b$, so $u'(\bar{c}_T)\geq u'(b)>0$: hence, $[\alpha_1 u'(\bar{c}_1)]/[\alpha_T u'(\bar{c}_T)]\leq[\alpha_1 u'(\bar{c}_1)]/[\alpha_T u'(b)]$. Using this, and (13) in (12), immediately yields (10).

(*Sufficiency*) Consider the following non-linear programming problem for T satisfying $1\leq T<\infty$.

$$\text{Maximize } \sum_{t=1}^{T}\alpha_t u(c_t)$$

(14) $$\text{subject to } c_{t+1}+x_{t+1}=y_{t+1};\ y_{t+1}=f(x_t) \quad \text{for} \quad t=0,\ldots,T-1$$

$$x_0 = \underset{\sim}{x};\ x_T \geq 0;\ x_{t+1},\ y_{t+1},\ c_{t+1} \geq 0 \qquad \text{for} \quad t = 0,\ldots, T-1.$$

The assumptions ensure that there is a unique solution to this problem for each T; call the solution $(x_{t+1}^T, y_{t+1}^T, c_{t+1}^T)_{t=0}^{T-1}$. The solutions for different T's have the following relations to each other. For each $t \geq 1$, $x_t^T \leq x_t^{T+1} \leq x_t^{T+2} \leq \cdots$, and there is a feasible program $(\tilde{x}, \tilde{y}, \tilde{c})$, called the *limit program* henceforth, from $\underset{\sim}{x}$, such that, for each $t \geq 1$, $x_t^T \to \tilde{x}_t$, $y_{t+1}^T \to \tilde{y}_{t+1}$, $c_{t+1}^T \to \tilde{c}_{t+1}$ as $T \to \infty$. (cf. Brock [1971], Theorem 2). We shall show that this limit program is weakly maximal when (10) holds. For this purpose, we need two lemmas. The reason for the lemmas can be explained simply. Lemma 3 ensures that the limit program is interior. This implies that, since the limit program is the limit of finite horizon optimal programs, it must satisfy the Euler conditions (see (16)).

If we can show that the limit program is efficient, then we can use the result of Brock [1971], (Corollary 3), viz., if a program satisfies the Euler conditions, and is efficient, then it is weakly maximal. This would conclude the proof of the theorem.

To show that the limit program is efficient, we consider two subcases given by Lemma 2, viz., either $\lim_{t\to\infty} \tilde{x}_t = 0$, or $\inf_{t \geq 0} \tilde{x}_t > 0$. If the first case holds it is well-known that the program is efficient. If the second case holds, we can apply the theorem of Cass [1972] to prove efficiency of the limit program if $\sum_{t=0}^{T} \tilde{\pi}_t \to \infty$ as $T \to \infty$. But to get this last result, from (10) and (16), we must make sure that $u'(\tilde{c}_t)$ is bounded above, that is, $\tilde{c}_t$ is bounded away from zero. And, this is precisely what Lemma 4 shows.

LEMMA 3. *For the limit program* $(\tilde{x}, \tilde{y}, \tilde{c})$ *from* $\underset{\sim}{x}$, *there is* $\hat{\varepsilon} > 0$, *such that* $\tilde{x}_t \leq b - \hat{\varepsilon}$ *for* $t \geq 0$.

PROOF OF LEMMA 3. Define $\hat{x}$, such that $0 < \hat{x} < b$, and $\left(1 - \frac{1}{2}m\right) \geq (\alpha_{t+1}/\alpha_t) f'(\hat{x})$ for all $t \geq 1$. By (A.1), this can be done. Now suppose the Lemma is false. Then, there is some $t' \geq 1$, such that $x_{t'} \geq \max\left[\hat{x} + \frac{1}{2}(b - \hat{x}),\ \underset{\sim}{x} + \frac{1}{2}(b - \underset{\sim}{x})\right]$. Hence, there is some T, such that the solution $(x_{t+1}^T, y_{t+1}^T, c_{t+1}^T)_{t=0}^{T-1}$ to problem (14) satisfies $x_{t'}^T \geq \hat{x}$, so that $(\alpha_{t'+1}/\alpha_{t'}) f'(x_{t'}^T) \leq (\alpha_{t'+1}/\alpha_{t'}) f'(\tilde{x}_{t'}) \leq (\alpha_{t'+1}/\alpha_{t'}) f'(\hat{x}) \leq \left(1 - \frac{1}{2}m\right) < 1$.

Since $(x_{t+1}^T, y_{t+1}^T, c_{t+1}^T)_{t=0}^{T-1}$ is a solution to (14), so for each t, such that $T-1 \geq t \geq 0$, the expression $\alpha_{t+1} u[f(x_t^T) - x] + \alpha_{t+2} u[f(x) - x_{t+2}^T]$ must be a maximum at $x = x_{t+1}^T$. By (U.4), the maximum must be at an interior point, i.e., $c_{t+1}^T > 0$ for $0 \leq t \leq T-1$, and $x_{t+1}^T > 0$ for $0 \leq t \leq T-1$. Hence, we have, for $0 \leq t \leq T-1$,

$$(15) \qquad [\alpha_{t+1} u'(c_{t+1}^T)]/[\alpha_{t+2} u'(c_{t+2}^T)] = f'(x_{t+1}^T).$$

Using this equation for $t = t'-1$, $u'(c_{t'}^T)/u'(c_{t'+1}^T) \leq \left(1 - \frac{1}{2}m\right) < 1$. Thus $c_{t'}^T > c_{t'+1}^T$, implying that $f(x_{t'-1}^T) - x_{t'}^T > f(x_{t'}^T) - x_{t'+1}^T$. Now, there are two possibilities to consider: (i) $x^T_{t'-1} < x^T_{t'}$, or (ii) $x^T_{t'-1} \geq x^T_{t'}$.

If (i) holds, then $x^T_{t'+1} > x^T_{t'} > \hat{x}$, so that the argument can be repeated to show that $x^T_{t'+2} > x^T_{t'+1}$. Similarly, repeating the argument for succeeding periods yields $x^T_{t'} < x^T_{t'+1} < \cdots < x^T_T$. But, clearly $x^T_T = 0$. This contradiction rules out (i).

If (ii) holds, then $x^T_{t'-1} \geq x^T_{t'} > \hat{x}$. So, repeating the analysis, $x^T_{t'-2} \geq x^T_{t'-1}$. Similarly, repeating the argument for each preceding period, we have $\underset{\sim}{x} = x^T_0 \geq x^T_1 \geq x^T_2 \geq \cdots \geq x^T_{t'-1} \geq x^T_{t'}$. But, we know that $x^T_{t'} > \underset{\sim}{x}$. This contradiction rules out (ii). Since (i) and (ii) were the only possibilities, Lemma 3 is established.

Returning to the main proof, note that $\tilde{c}_t > 0$ for some $t \geq 1$. Otherwise, if $\tilde{c}_t = 0$ for $t \geq 1$, then $\tilde{x}_{t+1} = f(\tilde{x}_t)$, and so $\tilde{x}_t \to b$ as $t \to \infty$, contradicting Lemma 3. Hence, by Corollary 1 of Brock [1971], $\tilde{c}_t > 0$ for all $t \geq 1$, and $\tilde{x}_t > 0$ for $t \geq 0$. Since $(\tilde{x}, \tilde{y}, \tilde{c})$ is the limit program, using (F.3), (U.2), and (15), we have

$$[\alpha_t u'(\tilde{c}_t)]/[\alpha_{t+1} u'(\tilde{c}_{t+1})] = f'(\tilde{x}_t) \quad \text{for} \quad t \geq 1. \tag{16}$$

By concavity of u and f, and (16), $(\tilde{x}, \tilde{y}, \tilde{c})$ is competitive.

We know from Lemma 1 that either (i) $\tilde{x}_t \to 0$ as $t \to \infty$, or (ii) $\inf_{t \geq 0} \tilde{x}_t > 0$. In case (i), $(\tilde{x}, \tilde{y}, \tilde{c})$ is clearly efficient, so by Corollary 3 in Brock [1971], it is weakly maximal. In case (ii), we need

LEMMA 4. *If, for the limit program* $(\tilde{x}, \tilde{y}, \tilde{c})$, $\inf_{t \geq 0} \tilde{x}_t > 0$, *then* $\inf_{t \geq 1} \tilde{c}_t > 0$.

PROOF OF LEMMA 4. We know that there is $\bar{k} > 0$, such that $\tilde{x}_t \geq \bar{k}$, for $t \geq 0$, and $\tilde{x}_t \leq b - \hat{\varepsilon}$ for $t \geq 0$. So, there is $k > 0$, such that $f(\tilde{x}_t) - \tilde{x}_t \geq k$, for $t \geq 0$. We note then that $\tilde{c}_t \leq \frac{1}{2} k$ cannot occur successively for T^* periods, where T^* is the largest integer, not greater than $[(2b/k) + 1]$. For, if it could occur, then $\tilde{x}_{t+1} = f(\tilde{x}_t) - \tilde{c}_{t+1} \geq \tilde{x}_t + \frac{1}{2} k$ for each of these periods. Thus, $\tilde{x}_{t+T^*} \geq \tilde{x}_t + \frac{1}{2} T^* k > \frac{1}{2} T^* k \geq b$, a contradiction.

Now, by (A.1), there is $\hat{Q} < \infty$, such that $(\alpha_{t+1}/\alpha_t) f'(\tilde{x}_t) \leq \hat{Q}$ for $t \geq 1$. Let $Q = \max(\hat{Q}, 1)$; and, $u'\left(\frac{1}{2} k\right) = M$. Suppose, contrary to the Lemma, $\tilde{c}_t \to 0$ along a subsequence. Then, there is t', such that $u'(\tilde{c}_{t'}) \geq Q^{T^*} M$. This means, by using (16), and the definition of Q that $u'(\tilde{c}_{t'+1}) \geq Q^{T^*-1} M$. Similarly, $u'(\tilde{c}_{t'+2}) \geq Q^{T^*-2} M$, and continuing this procedure, we finally have $u'(\tilde{c}_{t'+T^*}) \geq M$. Hence, $\tilde{c}_{t'}, \tilde{c}_{t'+1}, \ldots, \tilde{c}_{t'+T^*}$ are all $\leq \frac{1}{2} k$. But, this has already been shown to be impossible. Hence, $\inf_{t \geq 1} \tilde{c}_t > 0$, proving the Lemma.

Returning to the main proof again, we note that in case (ii), there is $\theta > 0$, such that $\tilde{c}_t > \theta$ for $t \geq 1$. Using (16) repeatedly, $[\alpha_1 u'(\tilde{c}_1)]/[\alpha_T u'(\theta)] \leq [\alpha_1 u'(\tilde{c}_1)]/[\alpha_T u'(\tilde{c}_T)] = \tilde{\pi}_T / f'(\underset{\sim}{x})$ for $T \geq 1$. Using (10), $\sum_{T=2}^{\tau} \tilde{\pi}_T \to \infty$ as $\tau \to \infty$. By Theorem 3 in Cass [1972], $(\tilde{x}, \tilde{y}, \tilde{c})$ is efficient. By Corollary 3 in Brock [1971], $(\tilde{x}, \tilde{y}, \tilde{c})$ is, therefore, weakly maximal.

4. A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF OPTIMAL PROGRAMS

Optimality is a stronger property than weak-maximality. Consequently, the necessary and sufficient condition for the existence of optimal programs should turn out to be stronger than (10). The result that is proved in this section is that the appropriate condition is the uniform boundedness of the sequence (α_t).

THEOREM 2. *Under (F.1)–(F.5), (U.1)–(U.4), and (A.1), an optimal program exists from* $\underset{\sim}{x} \in (0, b)$ *if and only if there is* $A < \infty$, *such that*

$$\alpha_t \leq A \qquad \textit{for} \quad t \geq 1 \tag{17}$$

PROOF. (*Sufficiency*) If (17) holds, then so does (10). Hence, there is a weakly-maximal program from $\underset{\sim}{x}$, call it $(\bar{x}, \bar{y}, \bar{c})$. Also, from the proof of Theorem 1, we know that $(\bar{x}, \bar{y}, \bar{c})$ must be interior and competitive, and identical to the limit program $(\tilde{x}, \tilde{y}, \tilde{c})$. We shall show that $(\bar{x}, \bar{y}, \bar{c})$ is optimal, when (17) holds.

By Lemma 1, either (i) $\bar{x}_t \to 0$ as $t \to \infty$, or (ii) $\inf_{t\geq 0} \bar{x}_t = v > 0$. In case (i), there is $t^* \geq 1$, such that $f'(\bar{x}_t) \geq 1$, for $t \geq t^*$, so that $\bar{p}_t$ is bounded above, and $\bar{p}_t \bar{x}_t \to 0$ as $t \to \infty$. Using the competitive conditions, the optimality of $(\bar{x}, \bar{y}, \bar{c})$ is immediate.

In case (ii), we apply Lemma 4 to get $\inf_{t\geq 1} \bar{c}_t > 0$. Hence, there is $\hat{V} < \infty$, such that $u'(\bar{c}_t) \leq \hat{V}$ for $t \geq 1$. Using (17), there is $V < \infty$ such that $\bar{p}_t \leq V$ for $t > 1$. Now suppose $(\bar{x}, \bar{y}, \bar{c})$ is not optimal. Then, there is a feasible program $(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{c}})$ from $\tilde{x}$, and $\mu > 0$, such that $\mu \leqq \sum_{t=1}^{T} \alpha_t [u(\bar{\bar{c}}_t) - u(\bar{c}_t)]$ for a subsequence T_s of T, and $\inf_{t\geqq 0} \bar{\bar{x}}_t \geq \frac{1}{2} v$.[6] Define $\delta_t = \bar{p}_t(\bar{\bar{x}}_t - \bar{x}_t) - \bar{p}_{t+1}(\bar{\bar{x}}_{t+1} - \bar{x}_{t+1}) - \alpha_{t+1}[u(\bar{\bar{c}}_{t+1}) - u(\bar{c}_{t+1})]$ for $t \geq 0$. Then, by the competitive conditions, $\delta_t \geq 0$ for $t \geq 0$, and $\sum_{t=1}^{T} \alpha_t [u(\bar{\bar{c}}_t) - u(\bar{c}_t)] = \bar{p}_T(\bar{x}_T - \bar{\bar{x}}_T) - \sum_{t=0}^{T-1} \delta_t$ for $T \geq 1$. Hence, for the subsequence T_s, $(\bar{x}_{T_s} - \bar{\bar{x}}_{T_s}) \geq (\mu/V) = \mu'$. Also, note that there are positive numbers $\hat{k}$, K such that for $\frac{1}{2} v \leq x \leq b$, we have $[-f''(x)] \geq \hat{k}$, and $f'(x) \leq K$. Hence, for the subsequence T_s, $\delta_{T_s} \geq \frac{1}{2}[-f''(\xi_{T_s})]\bar{p}_{T_s+1}[\bar{x}_{T_s} - \bar{\bar{x}}_{T_s}]^2 \geq \frac{1}{2}(\hat{k}/K)\mu\mu' = \delta$, say. Then, $\mu \leq \sum_{t=1}^{T_s} \alpha_t [u(\bar{\bar{c}}_t) - u(\bar{c}_t)] \leq \bar{p}_{T_s} (\bar{x}_{T_s} - \bar{\bar{x}}_{T_s}) - (s-1)\delta \leq Vb - (s-1)\delta$. For large s, the right-hand side becomes negative. This contradiction proves the optimality of

[6] If $(\bar{x}, \bar{y}, \bar{c})$ is not optimal, there is a feasible program from $\underset{\sim}{x}$, (x', y', c'), T_s, and $\mu > 0$, such that $\sum_{t=1}^{T_s} \alpha_t [u(c'_t) - u(\bar{c}_t)] \geq 2\mu$. Define a program $(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{c}})$ from $\underset{\sim}{x}$, by $\bar{\bar{x}}_t = \frac{1}{2}\bar{x}_t + \frac{1}{2}x'_t$; $\bar{\bar{y}}_{t+1} = f(\bar{\bar{x}}_t)$; $\bar{\bar{c}}_{t+1} = \bar{\bar{y}}_{t+1} - \bar{\bar{x}}_{t+1}$ for $t \geq 0$. Then $\inf_{t\geq 0} \bar{\bar{x}}_t \geq \left(\inf_{t\geq 0} \frac{1}{2}\bar{x}_t\right) = \frac{1}{2} v$ and $\sum_{t=1}^{T_s} \alpha_t [u(\bar{\bar{c}}_t) - u(\bar{c}_t)] \geq \mu$.

$(\bar{x}, \bar{y}, \bar{c})$.

(*Necessity*) If (x, y, c) is optimal, then, following the proof of Theorem 1, it is interior and competitive. Also, since it is weakly maximal, so by Corollary 2 of Brock [1971], it is identical to the limit program $(\tilde{x}, \tilde{y}, \tilde{c})$ of Section 3.

Suppose (17) is violated. Then, there is a subsequence t_s for which $\alpha_{t_s} \to \infty$ as $s \to \infty$. It can then be easily seen that x_t cannot converge to zero. For, if it did, π_t would be bounded below, and p_t would be bounded above. However, if $x_t \to 0$, then $c_t \to 0$ also, so that $u'(c_t) \to \infty$, and $p_{t_s} \to \infty$, a contradiction. Then, from Lemma 1, there is $\bar{\mu} > 0$, such that $x_t \geq \bar{\mu}$ for $t \geq 0$. And, from Lemma 4, there is $\bar{\theta} > 0$, such that $c_t \geq \bar{\theta}$, for $t \geq 1$. Let $\hat{\theta} = \frac{1}{2} f(\bar{\mu})(\bar{\theta}/b)$. Then, there are positive numbers, n, N, $\bar{M}$, such that for $\hat{\theta} \leq c \leq b$, we have $n \leq u'(c) \leq N$, and $[-u''(c)] \leq \bar{M}$. Now, define $\psi = \frac{1}{2} n\bar{\mu}$, and $\phi = (N/n)[1/f'(b)]\left[bN + \frac{1}{2}\bar{M}b^2\right]$. Choose a subsequence of t_s, call it t_r, such that (i) $t_{r+1} \geq t_r + 2$, and (ii) $\psi\alpha_{t_{r+1}} \geq \phi\alpha_{t_r}$ for $r \geq 1$. Since $\alpha_{t_s} \to \infty$ as $s \to \infty$, this is clearly possible. Finally, define $\lambda = \min\left[\frac{1}{2}, \frac{1}{2}(\bar{\theta}/b), (n\bar{\mu}/\bar{M}b^2)\right]$.

Now, construct a program $(\hat{x}, \hat{y}, \hat{c})$ in the following way: $\hat{x}_{t_r} = x_{t_r}(1-\lambda)$ and $\hat{x}_t = x_t$ for $t \neq t_r$; $\hat{y}_{t_r+1} = f(\hat{x}_{t_r})$ and $\hat{y}_t = y_t$ for $t \neq t_r + 1$; $\hat{c}_{t+1} = \hat{y}_{t+1} - \hat{x}_{t+1}$ for $t \geq 0$. To check for feasibility of $(\hat{x}, \hat{y}, \hat{c})$, note that $\hat{c}_{t_r} = c_{t_r} + \lambda x_{t_r} > 0$; and $\hat{c}_{t_r+1} \geq \{[c_{t_r+1}/f(x_{t_r})] - \lambda\} f(x_{t_r}) \geq [(\bar{\theta}/b) - \lambda] f(x_{t_r}) > 0$. For $t \neq t_r$, $t_r + 1$, $\hat{c}_t = c_t > 0$. Hence, $(\hat{x}, \hat{y}, \hat{c})$ is feasible.

To prove that (x, y, c) is not optimal, note that we have $\alpha_{t_1}[u(\hat{c}_{t_1}) - u(c_{t_1})]$ is positive; call it σ. Also, by straightforward calculations,[7] for $r \geq 1$, $\alpha_{t_r+1}[u(\hat{c}_{t_r+1}) - u(c_{t_r+1})] \geq -\alpha_{t_r+1}\left[\lambda bN + \frac{1}{2}\bar{M}\lambda^2 b^2\right] \geq -\alpha_{t_r}\lambda\phi$. Similarly, for $r \geq 1$, $\alpha_{t_{r+1}}[u(\hat{c}_{t_{r+1}}) - u(c_{t_{r+1}})] \geq \alpha_{t_{r+1}}\left[n\lambda\bar{\mu} - \frac{1}{2}\bar{M}\lambda^2 b^2\right] \geq \alpha_{t_{r+1}}\lambda\psi$. Recalling that the sequence t_r was chosen such that $\alpha_{t_{r+1}}\psi \geq \alpha_{t_r}\phi$ for $r \geq 1$, so we have, for all $r \geq 1$, $\sum_{t=1}^{t_r} \alpha_t[u(\hat{c}_t) - u(c_t)] \geq \sigma$ which proves that (x, y, c) is not optimal. This contradiction proves that (17) must hold if an optimal program exists.

5. AN ASYMPTOTIC STABILITY PROPERTY OF WEAKLY-MAXIMAL PROGRAMS

In this section, I shall assume that (10) holds, so that weakly maximal programs exist from initial inputs $\underset{\sim}{x} \in (0, b)$. I shall show, then, that these programs exhibit the following asymptotic stability property: they converge to each other in input levels. (This is often referred to, in the literature, as a "twisted turnpike" property). If optimal programs exist (i.e., (17) holds), then they will also exhibit the same property, since they are also weakly-maximal. I shall restrict the initial input levels to the closed interval $[a^*, b^*]$, where $0 < a^* < b^* < b$.

[7] This involves applying Taylor's expansion up to the second-derivative term, and then simplifying. The obvious details are omitted.

Before coming to the main result of the section, we need a lemma on finite-horizon programs, from different initial inputs. This lemma shows that if there are two finite horizon optimal programs with zero terminal input levels (i.e., which solve a problem like (14)), and one has a *larger initial input* level than the other, then it has a larger input and consumption level than the other *for every period*. The method used is similar to that of Brock [1971], Theorem 1, where a result is proved on two finite horizon optimal programs with the *same initial input* level, and *different terminal input* levels.

LEMMA 5. *Under* (*F*.1)–(*F*.5), (*U*.1)–(*U*.4), *and* (*A*.1), *if the solution to* (14), *with* $\underset{\sim}{x}$ *replaced by* $\bar{a}\in[a^*, b^*]$ *is* $(x_{t+1}^T(\bar{a}), y_{t+1}^T(\bar{a}), c_{t+1}^T(\bar{a}))_{t=0}^{T-1}$ *and the solution to* (14), *with* $\underset{\sim}{x}$ *replaced by* $\bar{b}\in[a^*, b^*]$ *is* $(x_{t+1}^T(\bar{b}), y_{t+1}^T(\bar{b}), c_{t+1}^T(\bar{b}))_{t=0}^{T-1}$, *and if* $\bar{a}\leq\bar{b}$, *then* $x_{t+1}^T(\bar{a})\leq x_{t+1}^T(\bar{b})$ *and* $c_{t+1}^T(\bar{a})\leq c_{t+1}^T(\bar{b})$, *for* $t=0, 1,\dots, T-1$.

PROOF OF LEMMA 5. To prove that $x_{t+1}^T(\bar{a})\leq x_{t+1}^T(\bar{b})$, suppose this is not true. Let t' be the first period when it is violated, i.e., $x_{t'+1}^T(\bar{a})>x_{t'+1}^T(\bar{b})$. Then, $c_{t'+1}^T(\bar{a})<c_{t'+1}^T(\bar{b})$. By the arguments used in Theorem 1, the solutions are interior, and hence they both satisfy (15). Hence $\alpha_{t'+2}u'(c_{t'+2}^T(\bar{a}))>\alpha_{t'+2}$ $u'(c_{t'+2}^T(\bar{b}))$, so $c_{t'+2}^T(\bar{a})<c_{t'+2}^T(\bar{b})$, and, by feasibility, $x_{t'+2}^T(\bar{a})>x_{t'+2}^T(\bar{b})$. So the argument can be repeated for all successive periods to obtain $x_T^T(\bar{a})>x_T^T(\bar{b})$, which contradicts the fact that $x_T^T(\bar{a})=x_T^T(\bar{b})=0$.

To prove that $c_{t+1}^T(\bar{a})\leq c_{t+1}^T(\bar{b})$, suppose this is not true. Let t' be the first period when it is violated, i.e., $c_{t'+1}^T(\bar{a})>c_{t'+1}^T(\bar{b})$. Using (15), and the fact that $x_{t'+1}^T(\bar{a})\leq x_{t'+1}^T(\bar{b})$, we have $c_{t'+2}^T(\bar{a})>c_{t'+2}^T(\bar{b})$. So the argument can be repeated for all successive periods to obtain $c_{t'+s}^T(\bar{a})>c_{t'+s}^T(\bar{b})$, for $s=1,\dots, T-t'$. But $x_{t'}^T(\bar{a})\leq x_{t'}^T(\bar{b})$, and $x_T^T(\bar{a})=x_T^T(\bar{b})$, which contradicts the fact that $(x_{t+1}^T(\bar{b}), y_{t+1}^T(\bar{b}), c_{t+1}^T(\bar{b}))$ is a solution to (14).

THEOREM 3. *Under* (*F*.1)–(*F*.5), (*U*.1)–(*U*.4), *and* (*A*.1), *given* $\varepsilon>0$, *there is* $T^*<\infty$, *such that if* $(x(\underset{\sim}{x}), y(\underset{\sim}{x}), c(\underset{\sim}{x}))$ *and* $(x(\underset{\sim}{x}'), y(\underset{\sim}{x}'), c(\underset{\sim}{x}'))$ *are weakly-maximal programs from* $\underset{\sim}{x}, \underset{\sim}{x}'\in[a^*, b^*]$, *then* $|x_t(\underset{\sim}{x})-x_t(\underset{\sim}{x}')|<\varepsilon$ *for* $t\geq T^*$.

PROOF. Consider the weakly maximal programs from a^* and b^*. Call them $(x(a^*), y(a^*), c(a^*))$ and $(x(b^*), y(b^*), c(b^*))$ respectively. Then, we know that these are the limit programs from a^* and b^*. Hence, by Lemma 5, $x_t(a^*)\leq x_t(b^*)$, and $c_{t+1}(a^*)\leq c_{t+1}(b^*)$ for $t\geq 0$. Now, by Lemma 1, either (i) $x_t(b^*)\to 0$ as $t\to\infty$, or (ii) $\inf_{t\geq 0} x_t(b^*)>0$. In case (i), there is T' such that $x_t(b^*)<\varepsilon$ for $t\geq T'$. Hence, for $t\geq T'$, $|x_t(b^*)-x_t(a^*)|<\varepsilon$.

In case (ii), we note that both the weakly maximal programs must be interior and competitive, and, hence, both must satisfy the set of Euler conditions (8). This means that, for $T\geq 2$,

$$\frac{u'(c_1(a^*))}{u'(c_1(b^*))}\cdot\frac{u'(c_T(b^*))}{u'(c_T(a^*))}=\frac{\prod_{s=0}^{T-1} f'(x_s(a^*))}{\prod_{s=0}^{T-1} f'(x_s(b^*))}\cdot\frac{f'(b^*)}{f'(a^*)} \tag{18}$$

Now, it is clear that $x_t(a^*)$ cannot converge to zero. For, then, there is t^*, such that $f'(x_t(a^*)) \geq 2f'(x_t(b^*))$ for $t \geq t^*$. Hence, the right-hand side of (18) will become unbounded for large t, while the left-hand side remains bounded. Thus, by Lemma 1, $\inf_{t \geq 0} x_t(a^*) = \hat{\mu} > 0$. Then, there are positive numbers L, $\hat{L}$ such that, for $\hat{\mu} \leq x \leq b$, $f'(x) \leq L$, and $[-f''(x)] \geq \hat{L}$. We claim, now, that, given $\varepsilon > 0$, there is T'', such that for $t \geq T''$, $|x_t(b^*) - x_t(a^*)| < \varepsilon$. If not, then there is a subsequence of periods, say t_r, such that this is violated. Then, since $[f'(x_{t_r}(a^*)) - f'(x_{t_r}(b^*))] = [x_{t_r}(a^*) - x_{t_r}(b^*)]\,[f''(\zeta_{t_r})] \geq \varepsilon\hat{L}$, so $[f'(x_{t_r}(a^*))/f'(x_{t_r}(b^*))] \geq [1 + (\varepsilon\hat{L}/L)]$. Then, as $r \to \infty$, the right-hand side of (18) becomes unbounded, while the left-hand side remains bounded, a contradiction which establishes our claim.

We let $T^* = \max(T', T'')$, and note then that in either case (i) or case (ii), $|x_t(a^*) - x_t(b^*)| < \varepsilon$, for $t \geq T^*$.

Now, consider the weakly maximal programs from $\underset{\sim}{x}$, $\underset{\sim}{x}'$ which both belong to $[a^*, b^*]$, but are otherwise arbitrary. Call these programs $(x(\underset{\sim}{x}), y(\underset{\sim}{x}), c(\underset{\sim}{x}))$ and $(x(\underset{\sim}{x}'), y(\underset{\sim}{x}'), c(\underset{\sim}{x}'))$. Since these are limit programs from $\underset{\sim}{x}$, $\underset{\sim}{x}'$, so by Lemma 5, we know that $x_t(a^*) \leq x_t(\underset{\sim}{x}) \leq x_t(b^*)$, and $x_t(a^*) \leq x_t(\underset{\sim}{x}') \leq x_t(b^*)$ must hold for $t \geq 0$. Hence, for $t \geq T^*$, $|x_t(\underset{\sim}{x}) - x_t(\underset{\sim}{x}')| \leq |x_t(a^*) - x_t(b^*)| < \varepsilon$, which proves the Theorem.[8]

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[8] Notice that it is immediate from this section that weakly-maximal programs also converge to each other in terms of output and consumption levels.

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